**Fractals 12**

**Permutation Entropy**

Shannon’s and other classical entropy measures neglect temporal relationships between the values of the time series, and fail to adequately capture the correlational structure in its behavior [1]. To overcome this problem Bandt and Pompe [2] introduced the new complexity measure Permutation entropy, a simple and robust method that takes into account time causality by comparing neighboring values in a time series. The continuous time series is mapped into symbolic sequence that arises naturally from the time series, by comparing the order of neighboring relative values. Permutation entropy algorithm proceeds as follows. For a given time series we form  -dimensional vectors  where  is the embedding dimension (the number of samples included in each motif) and  is the time lag (the number of samples spanned by each section of the motif). For   different numbers in each sequence, there will be  possible ordinal patterns , which are also called permutations. For each vector ordinal pattern is defined as the permutation  of which fulfills . For each  we determine the relative frequency where denote its frequency in the series of vectors . The permutation entropy of order is defined as



where the sum runs over all  permutations  of order . For this is the information contained in comparing  consecutive values of the time series. It is clear that where the lower bound is attained for an increasing or decreasing sequence of values (only one permutation appears), and the upper bound for a completely random system where all  possible permutations appear with the same probability. The time series presents some sort of dynamics when. The optimal  and  strongly depend on the system. In order to get good statistics it is typically recommended to choose maximum  according to [3].

***Example***: Let us take a series with values:  . We choose e , and construct the series of sequences of vectors {(3,6,8),(6,8,9),(8,9,5),(9,5,10), (5,10,2)}. The corresponding series of permutations is {(012),(012),(201),(102), (201)}. There are  possible permutations (012,021,102,120,201,210), but only three permutations 012 (two times), 201 (two times) e 102 (one time) appear in a series. The permutation entropy is calculated as



Permutation entropy method and its multiscale generalization were applied to study various phenomena such as in physiology [4,5], engineering [6,7], geophysics [8] climatology [9], hydrology [10,11] and finances [12].



***Forbiden patterns***

The study of the order patterns has been proposed as a technique for evaluating the determinism of a given time series [[13]](http://www.sciencedirect.com/science/article/pii/S0378437109002593#b40). Every group of  adjacent and overlapped values of the time series has an associated permutation . This permutation will be one of the  possible permutations. If the series has a random behavior, any permutation can appear. The probability distribution of the ordinal patterns should be uniform because any permutation has the same probability of appearance when the data set is long enough to exclude statistical fluctuations. Nevertheless, when the series corresponds to a chaotic variable, there are some patterns that cannot be found due to the underlying deterministic structure. They are the so-called forbidden patterns. The existence of forbidden patterns indicates an underlying deterministic behavior [14]. For correlated stochastic processes the existence of a non-observed ordinal pattern does not qualify as “*forbidden”,* only as “missing” due to the time series finite length. The concept of forbidden/missing patterns was used to study financial data [14,15], physiological signals [16] and hydrological time series [17]

***Complexity-entropy causality plane***

Lamberti et al. [[18]](http://www.sciencedirect.com/science/article/pii/S0378437110000397#b38) introduced a statistical complexity measure (SCM) that provides important additional information regarding the peculiarities of the underlying probability distribution, not already detected by the entropy. For permutation probability distribution this statistical complexity measure is defined as



where



is normalized permutation entropy with ;  for completely random series when all possible  permutations appear with same probability; for increasing or decreasing series when only one permutation appears. For series with more complex order dynamics . The quantity  is called disequilibrium and it is defined as

,

Where is Jensen–Shannon divergence (that quantify the difference between two probability distributions), is the uniform distributions (when all permutations are equally probable) and



is a normalization constant equal to the maximum possible value of  (when all components of  are equal to zero except one).

This new quantity  not only quantify randomness but also the presence of correlations among components of dynamic system and thus provides more information than the entropy  [19,20]. Naturally  is different than zero only when there exist more likely permutations among the accessible ones. The opposite extremes, perfect order and maximal randomness (i.e. a periodic sequence and a fair coin toss) possess no structure; these systems are too simple and have zero statistical complexity. In between these two special cases, there exist a wide range of possible degrees of physical structure that should be reflected in the features of the underlying probability distribution and consequently in the values of . Also   is not a trivial function of the entropy because it depends on two different probabilities distributions, the one associated to the system under analysis, , and the uniform distribution, .It was shown that for a given  value, there exists a range of possible SCM values between a minimum  and a maximum .  Based on this characteristic Rosso et al. [21] introduced a diagram  versus called *Complexity-entropy causality plane* which is obtained with the permutation entropy of the system  in the horizontal axis and the permutation statistical complexity in the vertical one. The term causality  takes into consideration that the temporal correlation between successive samples is included in the permutation probability distribution used to estimate  and . This representation space is particularly useful to discriminate between chaotic systems and stochastic processes, locating them at different planar positions  [21]. *Complexity-entropy causality plane* was used in finances [22,23], hydrology [11]and music [24].

**Causal Fisher Shannon plane (FS)**

Fisher Shannon plan (FS) was introduced by Vignat and Bercher [25], with Shannon entropy on horizontal axes and Fisher information measure on vertical axes. The “causal” version [26], includes Permutation entropy (x- axes) and Fisher information measure (y-axes). Both quantifiers are calculated for Bandt-Pompe probability distribution. This version of FS was shown useful in the analysis of data in physiology [27], finances [28], hydrological processes [29], ecology [30] and pattern recognition [31]. Permutation entropy was defined at (1), what follows is the description of Fisher information measure (FIM).

For a continuous random variable with probability density function , Fisher Information measure (FIM) is defined as [54]

which should not be confused with the Fisher information of a distribution parameter. For a discrete distribution Fisher information measure is formulated as [39]

where and are probabilities from that correspond to consecutive states, and normalization constant is taken as if or , and otherwise. Opposite of Shannon entropy behavior, when the knowledge about the system is complete, FIM assumes the maximal value , while for a uniform distribution when the information about the system is minimal, While entropy and complexity (defined in equations (1) and (2) respectively), are global information measures, FIM is a local quantifier which depends on the ordering of probabilities of consecutive states, In Bandt-Pompe distribution the “states” are ordinal patterns that can be ordered in different ways each producing different FIM values. In this work we chose the method of lexicographic ordering (or Lehmer coding), which was shown to well differentiate different types of processes in the FS plane [58]. For example, for vector of dimension , there are six (3!) possible patterns (states) which are ordered as ; ; ; ; ; .

Fisher Shannon plane can be defined for continuous data [32, 33].

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